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Scalar-Tensor Gravity on a Gauss-Bonnet Brane World

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ABSTRACT: The effective four-dimensional, linearised gravity for a brane world model with higher order curvature terms and a bulk scalar field is analysed. Large and small distance gravitational laws are derived. The model has a single brane embedded in a five-dimensional bulk spacetime, and the scalar field represents the dilaton or a moduli field. The quadratic, Gauss-Bonnet curvature term (and corresponding higher kinetic terms for the scalar) is also included in the bulk action. It is particularly natural to include such terms in a brane world model. Boundary terms and junction conditions for the higher order terms are given.

The extra terms allow additional solutions of the field equations, which give better agreement with observational constraints. Brans-Dicke gravity is obtained on the brane. The scalar and tensor perturbations are affected differently by the higher gravity terms, and this provides a way for the scalar modes to be suppressed relative to the tensor ones. Another new (but less useful) feature is the appearance of instabilities for some parameter ranges.

KEYWORDS: eld, ctg.

Contents

1. Introduction

There has been considerable interest in the idea [1] that we may live on 3+1 dimensional brane embedded in a 4+1 dimensional bulk spacetime. Despite the extra dimension, it is still possible for the effective gravitational theory on the brane to closely resemble that which is observed in our universe. The models we will be considering in this paper resemble the well known Randall-Sundrum II brane world model [2]. In this scenario the graviton spectrum has a zero mode which, due to the warping of the bulk space, is localised on the brane. This is interpreted as the four-dimensional graviton. At large distance scales the effects of the other modes are suppressed, and something close to four-dimensional general relativity is obtained for brane-based observers [3]. Gravity will have unconventional behaviour at short distances, and so to be compatible with gravity experiments, 'short' needs to of order $0.1\,\mathrm{mm}$ or less.

The brane world scenario is loosely motivated by string theory. However, if such a model were actually derived from string theory, the bulk space would contain not only gravity, but many other fields as well. The simplest example of this would be a scalar field such as the dilaton, or a moduli field coming from the compactification of other extra dimensions. One of the purposes of this paper is to investigate the implications of bulk scalar fields for brane world gravity. For simplicity I will include just one such field, and consider only a special class of solutions for which the field equations can be easily solved.

Scalar-tensor gravity has previously been studied in a brane world context [4], although most previous work has considered only the graviton modes. However, given the strong

constraints placed on scalar modes by solar system measurements [5], we should not ignore them if we are studying a brane world which is supposed to represent our universe.

Another natural extension of the usual brane world model is to add higher order curvature terms to the action. One such term is the quadratic Gauss-Bonnet (or Lanczos [6]) term

$$\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{ab}R^{ab} + R^{abcd}R_{abcd} \ . \tag{1.1}$$

Like the Einstein-Hilbert action this gives rise to field equations which are divergence free, and do not contain any third or higher order derivatives of the metric. They therefore have all the desirable properties of the Einstein equation (such as a ghost-free vacuum and conservation of energy momentum) [7]. Note that this is the only quadratic curvature term which has all these properties. In four dimensions its contribution to the field equations is trivial, and so it is usually ignored.

The implications of this term for brane world gravity have been discussed in the literature [8, 9, 10, 11, 12]. Note that the derivation of the brane junction conditions is more complicated than in the conventional brane world [9, 13], and this led to some subtle mistakes in early works [8].

Like the branes themselves, higher order curvature terms (and corresponding higher order scalar kinetic terms) are also motivated by string theory. It is therefore very natural to consider them in brane world models. For simplicity, we will only consider the Gauss-Bonnet term and the corresponding scalar field terms in this paper (see ref. [10] for a discussion of even higher order curvature terms in a brane world context).

The layout of the paper is as follows: In section 2 the details of the model are discussed. Solutions of the field equations are described, with and without the higher order terms in the action. Not surprisingly, the addition of the higher order terms produces extra solutions to the field equations. For want of a better name we will call these the 'Gauss-Bonnet branches', and the other solutions the 'Einstein branch'. Linearised brane world gravity is reviewed in section 3. Linear perturbations of the 'Einstein' solutions are considered in section 4, and the more interesting and better behaved 'Gauss-Bonnet' solutions are analysed in section 5. Several of the solutions are shown to be unstable, although some of these instabilities can be removed by adding appropriate terms to the brane part of the action. These stabilised solutions are discussed in section 6. The last section contains a summary of the results. The bulk field equations and junction conditions (together with corresponding boundary terms) are listed in the appendix.

2. Bulk Solutions

We will consider a five-dimensional toy model with a single, Z_2-symmetric brane and one bulk scalar field, Φ, which is conformally coupled to gravity. We will be working in the Jordan frame (in a string theory context, and when Φ is the dilaton, this is known as the string frame). We start by considering the first order action

$$S_1 = \frac{M^3}{2} \int d^5x \sqrt{-g} \, e^{-2\Phi} \left\{ R - 4\omega(\nabla\Phi)^2 - 2\Lambda \right\} - M^3 \int d^4x \sqrt{-h} \, e^{-2\Phi} \left\{ 2K + T \right\} \ ,$$

$$\tag{2.1}$$

where $h_{ab} = g_{ab} - n_a n_b$ is the induced metric on the brane and $K_{ab} = h^c{}_a \nabla_c n_b$ is the extrinsic curvature. M is the five-dimensional Planck mass, Λ is the bulk cosmological constant, and T is the brane tension. The brane is being treated as a boundary of the two halves of the bulk spacetime, and the Gibbons-Hawking boundary term has been included to give a consistent action [14]. In this paper we take the brane normal to be pointing away from the brane on both sides of the brane. This is an unconventional choice, but it is easier to visualise.

The general second order action is

$$
S_2 = \frac{M}{2} \int d^5 x \sqrt{-g}\, e^{-2\Phi} \left\{ c_1 \mathcal{L}_{\text{GB}} - 16 c_2 G_{ab} \nabla^a \Phi \nabla^b \Phi + 16 c_3 (\nabla \Phi)^2 \nabla^2 \Phi - 16 c_4 (\nabla \Phi)^4 \right\}
$$
$$
- M \int d^4 x \sqrt{-h}\, e^{-2\Phi} \left\{ \frac{4}{3} c_1 \left(3 K K_{ab} K^{ab} - 2 K^a{}_b K^{bc} K_{ac} - K^3 - 6 \widehat{G}^{ab} K_{ab} \right) \right.
$$
$$
\left. - 16 c_2 (K_{ab} - K h_{ab}) D^a \Phi D^b \Phi + \frac{16}{3} c_3 \left((\partial_n \Phi)^3 + 3 \partial_n \Phi (D\Phi)^2 \right) \right\} , \quad (2.2)
$$

where $\partial_n \equiv n^a \partial_a$. The caret denotes tensors with correspond to the induced metric h_{ab}, and D_a is the corresponding covariant derivative.

When the scalar field is included there are a total of four possible second order terms which produce suitable field equations (no higher than second derivatives, etc.). For the action to be consistent, we must include the Gauss-Bonnet equivalent of the Gibbons-Hawking boundary term [13, 15], as well as the corresponding Φ dependent terms (see appendix).

We see that the boundary parts of the above actions include terms which are first and third order in derivatives. It is therefore natural to also include an induced gravity action

$$
S_{\text{Ind}} = \frac{M^2}{2} \int d^4 x \sqrt{-h}\, e^{-2\Phi} \left\{ b_1 \widehat{R} + b_2 (D\Phi)^2 \right\} , \quad (2.3)
$$

which is second order in derivatives. Although we will consider this term, we will be mainly interested the implications of the other parts of the action, (2.1) and (2.2).

Scalar-tensor theories of gravity are also studied in the Einstein frame, which is related to the Jordan frame by the conformal transformation. The transformation is chosen so that Φ-dependent factor in front of Einstein-Hilbert part of the action is cancelled. For a brane world this is either $g_{ab} \rightarrow e^{4\Phi/3} g_{ab}$ or $g_{ab} \rightarrow e^{2\Phi} g_{ab}$, depending on whether we are considering the bulk (2.1) or brane (2.3) part of the action. So we see that there are actually two Einstein frames for a brane world.

The values of the coefficients ω, c_i and b_i will be determined by the fundamental theory from which the actions (2.1–2.3) are derived. One of the simplest possibilities is to suppose that Φ is a moduli field arising from a toroidal compactification of a higher dimensional space. Consider a $(5 + N)$-dimensional theory, whose bulk action contains only the Einstein-Hilbert and Gauss-Bonnet terms. Take the metric to have the form

$$
ds_{5+N}^2 = g_{ab}(x) dx^a dx^b + e^{-4\Phi(x)/N} \eta_{AB} dX^A dX^B \quad (2.4)
$$

and then compactify the last N dimensions (we have made the simplifying assumption that these extra dimensions only have one degree of freedom) [12]. Comparison of field

equations reveals that such a theory will be equivalent to that obtained from the above actions if $\omega = -1 + N^{-1}$,

$$c_2 = -\omega c_1 \ , \quad c_3 = (2\omega + 1)\omega c_1 \ , \quad c_4 = -(2\omega + 1)\omega^2 c_1 \tag{2.5}$$

and $b_2 = -4\omega b_1$. In the $N \to \infty$ limit, Φ can be interpreted as the dilaton for a theory with a high degree of symmetry [12]. For the rest of this paper we will assume the above relation between c_i, b_i and ω, and take $0 \geq \omega \geq -1$. For later convenience we define $\alpha = c_1/M^2$ and $\beta = b_1/M$. Both of these parameters will be taken to be positive.

Combining all the above contributions to the action (2.1–2.3), and including some brane matter gives the total action

$$S = S_1 + S_2 + S_{\mathrm{Ind}} - \int d^4 x \sqrt{-h}\, \mathcal{L}_{\mathrm{mat}} \ . \tag{2.6}$$

The brane matter Lagrangian $\mathcal{L}_{\mathrm{mat}}$ will be treated as a small perturbation to the background solution. For now we will set it to zero.

In this paper we will be interested in perturbations of non-singular brane world solutions of the form

$$ds^2 = e^{-2k|z|} dx^\mu dx^\nu \eta_{\mu\nu} + dz^2 \ , \tag{2.7}$$

with $\Phi = \phi_0 - \sigma|z|$. The single brane is located at $z = 0$, and the bulk is taken to be Z_2-symmetric. Solutions of this form are the simplest extension of the usual Randall-Sundrum model [2]. Previous work [16, 17, 18] on such solutions in Gauss-Bonnet brane worlds has used different coefficients (c_i) for the terms in bulk action (2.2), although this did not lead to qualitatively different results. Other, more general classes of solutions exist [12, 17], but we will not consider them here.

There are problems with the above solution if $\sigma < 0$. Although (in the Jordan frame) there are no curvature singularities, the coupling of matter to gravity (which is proportional to e^Φ) will be divergent as $z \to \infty$. If $k > 0$ the time taken (as observed on the brane) for a massless particle to travel from this singularity to the brane will be infinite. However it is still possible for signals from points arbitrarily close to the singularity (and with arbitrarily high coupling) to reach the brane in finite time. The low energy action we are using will not be valid near the singularity, and so the physics of our brane universe would be dominated by effects that we have not included in our model. In the Einstein frames (bulk and brane), the curvature at this point becomes singular, and furthermore the singularity will then be at a finite proper distance from the brane. Therefore, in order to have a consistent single brane model, we will only consider solutions with $\sigma \geq 0$. e^Φ is then bounded above by e^{ϕ_0}, its value on the brane.

Considering only the lowest order action (2.1), the bulk field equations (see appendix) are solved by

$$k = -2(\omega + 1)\sigma \ . \tag{2.8}$$

The value of σ will be related to the bulk cosmological constant. We see that (for the $\sigma > 0$, $\omega \geq -1$ solutions we are considering) the sign of warp factor is the opposite of that

in the usual brane world model. As we will see in section 4, the negative warp factor does not give a suitable effective theory of gravity on the brane.

When the higher order terms (2.2) are included, the above solution is still valid, and (if $\omega \neq 0$) two new branches of solutions appear

$$k = -\omega\sigma \pm \sqrt{\frac{1}{12\alpha} - \frac{\omega(\omega+2)}{3}\sigma^2} \tag{2.9}$$

(if $\omega = 0$ then $k^2 = 1/[12\alpha]$, the field equations are degenerate and Φ is undetermined). It is possible for these solutions to have positive warp factor, suggesting that a viable effective gravitational theory could be obtained on the brane. These new solutions will be referred to as the 'Gauss-Bonnet branches', and the other solution as the 'Einstein branch'. It is perhaps worth noting that for a more general choice of the coefficients in the action (2.2), the 'Einstein' solution will not still be valid when the higher order gravity terms are switched on (see e.g. [16, 18]).

3. Linearised Brane Gravity

We will start by reviewing linearised gravity for a brane world without scalar fields. By taking $\mathcal{L}_{\mathrm{mat}}$ to be small, a perturbative analysis can be used to determine the approximate effective gravitational law on the brane. We will use a gauge in which the brane is kept straight, even when matter is present [19, 20].

The general perturbed metric can be written as

$$ds^2 = e^{-2k|z|}(\eta_{\mu\nu} + \gamma_{\mu\nu})dx^\mu dx^\nu + 2v_\mu dx^\mu dz + (1+\psi)dz^2 . \tag{3.1}$$

The quantities $\gamma_{\mu\nu}$, v_μ and ψ are all small. The perturbed brane normal is then $n_a = \delta_a^z \operatorname{sign}(z)(1+\psi/2)$. We use the metric convention $\eta_{\mu\nu} = \operatorname{diag}(-1,1,1,1)$.

The lapse function, ψ, and the shift vector, v_μ, are determined by solving the components of the field equations which are perpendicular to the brane. If the brane does not bend, as we are requiring, these will not depend on the brane matter. For the Gauss-Bonnet-Randall-Sundrum model ($k > 0$) they are solved (for $z > 0$) by

$$\psi = -\frac{1}{4k}\partial_z\gamma \tag{3.2}$$

$$v_\mu = -\frac{1}{8k}\partial_\mu\gamma + B_\mu \tag{3.3}$$

$$\partial^\mu\bar{\gamma}_{\mu\nu} = 0 \tag{3.4}$$

where $\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - (1/4)\gamma\eta_{\mu\nu}$, $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$, and B_μ satisfies $\partial^\mu B_\mu = 0$ and $\Box_4 B_\mu = 0$. We can use gauge freedom to set $B_\mu = 0$.

For the above choice of gauge the remaining bulk field equations reduce to

$$(1 - 4\alpha k^2)\left(\partial_z^2 - 4k\partial_z + e^{2kz}\Box_4\right)\bar{\gamma}_{\mu\nu} = 0 \tag{3.5}$$

with $\Box_4 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. There is no bulk equation for γ. The remaining gauge freedom can used to choose its behaviour there (but not its value on the brane) [19].

Note that the above equation is free from third or higher order derivatives of $\bar{\gamma}_{\mu\nu}$, and is similar to linearised Einstein gravity in this respect. This follows from the fact that the Gauss-Bonnet term was specifically constructed to give a Einstein-like theory of gravity. If $1 - 4\alpha k^2 < 0$ the $\bar{\gamma}_{\mu\nu}$ kinetic term in the linearised effective action will be negative, and so the theory will have bulk ghosts [10, 12]. Therefore if the theory is to be stable on the quantum level, k must be less than $1/\sqrt{4\alpha}$.

The junction conditions at $z = 0$ give

$$2(1 - 4\alpha k^2)\partial_z\bar{\gamma}_{\mu\nu} + (\beta + 8\alpha k)\Box_4\bar{\gamma}_{\mu\nu} = -\frac{2}{M^3}\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\} \tag{3.6}$$

and

$$(1 + \beta k + 4\alpha k^2)\Box_4\gamma = \frac{4k}{3M^3}S \tag{3.7}$$

where $S_{ab} = 2\delta\mathcal{L}_{\text{mat}}/\delta h^{ab} - h_{ab}\mathcal{L}_{\text{mat}}$ is the energy-momentum tensor for the brane matter.

Switching to Fourier space, the bulk graviton equation (3.5) is solved, for spacelike momenta ($p^2 > 0$), by

$$\bar{\gamma}_{\mu\nu}(p, z) \propto e^{2k|z|}K_2\left(pe^{k|z|}/k\right) . \tag{3.8}$$

We do not use the other solution as it diverges as $z \to \infty$. Substituting this into the junction conditions allows $\gamma_{\mu\nu}$, and hence the effective gravitational laws, to be determined on the brane.

Define $\mathcal{G}_{\mu\nu}$ to be the Einstein tensor corresponding to the perturbation $\gamma_{\mu\nu}$. At large distances ($p \ll k$) an asymptotic expansion of the solutions shows that, to leading order in p, $\mathcal{G}_{\mu\nu} \approx M_{\text{Pl}}^{-2}S_{\mu\nu}$ with

$$M_{\text{Pl}}^2 = \frac{1 + k\beta + 4\alpha k^2}{k}M^3 . \tag{3.9}$$

Therefore general relativity with effective four dimensional Planck mass M_{Pl} is obtained.

If at least one of α and β is non-zero, the linearised brane gravity at short distances ($p \gg k$) will instead be described by

$$\mathcal{G}_{\mu\nu} - 2(\eta_{\mu\nu}\Box_4 - \partial_\mu\partial_\nu)\tilde{\varphi} \approx M_{\text{Pl}}^{-2}S_{\mu\nu} \tag{3.10}$$

$$-2\Box_4\tilde{\varphi} \approx M_\phi^{-2}S \tag{3.11}$$

where

$$M_{\text{Pl}}^2 = (\beta + 8\alpha k)M^3 \tag{3.12}$$

$$M_\phi^2 = \frac{3(1 + \beta k + 4\alpha k^2)}{4(1 - 4\alpha k^2)}M_{\text{Pl}}^2 , \tag{3.13}$$

and $\varphi \propto \gamma$. This is linearised Brans-Dicke gravity. M_ϕ^{-2} gives the coupling strength for the scalar degree of freedom. If Brans-Dicke theory is to be compatible with gravitational measurements of the solar system, we need $M_{\text{Pl}}^2/M_\phi^2 < 10^{-3}$ [5].

If $\alpha = \beta = 0$ the short distance gravity will be very non-standard, and this leads to strong constraints on the brane world parameters ($1/k \lesssim 0.1\,\text{mm}$). In contrast, the short distance constraints on scalar-tensor gravity theories, and hence the above brane world, are quite weak ($1/k \lesssim 100\,\text{km}$ if $4\alpha k^2$ is near to 1 [11]).

4. Einstein Branch

A similar analysis to that in the previous section can be applied to the brane world solutions of section 2. We use the perturbed metric (3.1), and take $\Phi = \phi_0 - \sigma|z| + \varphi$, with φ small. As before, v_μ and ψ are determined by requiring that the brane remains at $z = 0$. The scalar field perturbations mix in with the trace of the metric perturbations, and this requires a slightly different choice of gauge.

The 'Gauss-Bonnet' branches will be considered in the next section. We will begin with the 'Einstein' branch, which has $k = -2(\omega + 1)\sigma$. We are interested in the parameter range $0 \leq \omega \leq -1$, and so in contrast to the Randall-Sundrum model, this solution will have negative warp factor. Not surprisingly the resulting brane gravity is significantly different to that in the previous section.

It is useful to split $\gamma_{\mu\nu}$ into tensor and scalar parts

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \frac{1}{4}\gamma\eta_{\mu\nu} + \frac{4}{3}\left(\frac{1}{4}\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)\chi \tag{4.1}$$

with

$$\chi = \frac{8k\varphi - \sigma\gamma}{4(2k - \sigma)} . \tag{4.2}$$

Requiring that the brane does not bend gives

$$\psi = \frac{2}{\sigma}\partial_z(\chi - \varphi) \tag{4.3}$$

$$v_\mu = \frac{1}{\sigma}\partial_\mu(\chi - \varphi) + B_\mu \tag{4.4}$$

$$\partial^\mu\bar{\gamma}_{\mu\nu} = 0 . \tag{4.5}$$

Gauging away B_μ, the remaining bulk equations (for $z > 0$) reduce to

$$\mu_0\left(\partial_z^2 - 2(2k - \sigma)\partial_z + e^{2kz}\Box_4\right)\bar{\gamma}_{\mu\nu} = 0 \tag{4.6}$$

$$4\mu_0(\omega + 4/3)\left(\partial_z^2 - 2(2k - \sigma)\partial_z + e^{2kz}\Box_4\right)\chi = 0 \tag{4.7}$$

where $\mu_0 = 1 - 4\alpha(2\sigma - k)(\sigma - k)$. Again there is the possibility of ghosts if the bulk curvature is too high. To avoid this we need $\mu_0 > 0$ (and $\omega > -4/3$).

Taking $\mathcal{L}_{\text{mat}}$ to be independent of Φ, the brane junction conditions imply

$$2\mu_0\partial_z\bar{\gamma}_{\mu\nu} + (\beta - 8\alpha[2\sigma - k])\Box_4\bar{\gamma}_{\mu\nu} = -\frac{2\sqrt{\alpha}}{M_\star^2}\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\} \tag{4.8}$$

$$2\mu_0\partial_z\chi + (\beta - 8\alpha[2\sigma - k])\Box_4\chi = -\frac{\sigma\sqrt{\alpha}}{(2\sigma - 3k)M_\star^2}S \tag{4.9}$$

and

$$(1 + [4\alpha(2\sigma - k) - \beta][\sigma - k])\Box_4(\chi - \varphi) = \frac{\sigma k\sqrt{\alpha}}{2(2\sigma - 3k)M_\star^2}S \tag{4.10}$$

where $M_\star^2 = M^3 e^{-2\phi_0}\sqrt{\alpha}$.

For spacelike momenta the solution of the bulk equations (4.6,4.7) which vanishes as $z \to \infty$ is

$$e^{(2k-\sigma)z} I_{2-\sigma/k} \left(-pe^{kz}/k \right) , \tag{4.11}$$

unless $k = 0$ ($\omega = -1$), in which case it is

$$\exp\left(-\left[\sigma + \sqrt{\sigma^2 + p^2} \right] z \right) . \tag{4.12}$$

Substituting in the bulk solution and setting $z = 0$, we find that the left hand side of the graviton equation (4.8) is equal to

$$-\left\{ 2\mu_0 \frac{p I_{1-\sigma/k}(-p/k)}{I_{2-\sigma/k}(-p/k)} + (\beta - 16\alpha[2 + \omega]\sigma)p^2 \right\} \bar{\gamma}_{\mu\nu} \tag{4.13}$$

for $\omega > -1$ and

$$-\left\{ 2\mu_0(\sqrt{\sigma^2 + p^2} + \sigma) + (\beta - 16\alpha\sigma)p^2 \right\} \bar{\gamma}_{\mu\nu} \tag{4.14}$$

for $\omega = -1$. The left hand side of the scalar equation (4.9) will have a similar form.

If the above term in brackets vanishes for some $p > 0$, then eqs. (4.8) and (4.9) will have non-trivial solutions when $\mathcal{L}_{\text{mat}} = 0$. In this case the vacuum will have tachyon modes, and so will be classically unstable. If the induced terms in the action (2.3) are absent, or small ($\beta < 16\alpha[2 + \omega]$), this is always the case. Hence the addition of the higher order gravity terms has destabilised the solution.

If the quadratic curvature terms are absent, the above brane world has a negative tension brane, suggesting that the system is unstable. In fact it is stable, but only because of the Z_2-symmetry. It is therefore not at all surprising that adding extra terms to the action (even if α is tiny) removes the stability. Similar effects were noted for the brane world models discussed in ref. [12], and also for the two brane Randall-Sundrum model [21]. The problem could be avoided by taking β to be large. This will be discussed in section 6.

5. Gauss-Bonnet Branches

We will now consider the new branches of solutions which appear when the quadratic curvature terms are included in the action. These have the bulk, background solution

$$k = -\omega\sigma \pm \sqrt{\frac{1}{12\alpha} - \frac{\omega(\omega + 2)}{3}\sigma^2} . \tag{5.1}$$

We will only consider $-1 \leq \omega < 0$, since the field equations are degenerate when $\omega = 0$. As in the previous section $\gamma_{\mu\nu}$ should be split into scalar and tensor parts. We use the same gauge as before (4.3–4.5), but this time take

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \frac{1}{4}\gamma\eta_{\mu\nu} + \frac{4}{3}c_\chi \left(\frac{1}{4}\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4} \right) \chi \tag{5.2}$$

where the field χ is now defined by

$$\chi = \frac{3(8k\varphi - \sigma\gamma)}{4(6k + c_\chi\sigma)} , \tag{5.3}$$

with

$$c_\chi = \frac{\omega\sigma(3k + 2(2\omega + 1)\sigma)}{k(k + \omega\sigma)} \ . \tag{5.4}$$

Using the above expressions, the bulk equations reduce to

$$\mu_\gamma \left(\partial_z^2 - 2(2k - \sigma)\partial_z + f_\gamma^2 e^{2kz}\Box_4\right) \bar{\gamma}_{\mu\nu} = 0 \tag{5.5}$$

and

$$\mu_\chi \left(\partial_z^2 - 2(2k - \sigma)\partial_z + f_\chi^2 e^{2kz}\Box_4\right) \chi = 0 \tag{5.6}$$

where

$$f_\gamma^2 = \frac{k - (1 - \omega)\sigma}{\omega\sigma + k} \tag{5.7}$$

$$f_\chi^2 = \frac{3k}{3k + 2(2\omega + 1)\sigma} \tag{5.8}$$

$$\mu_\gamma = 8\alpha(k + 2[\omega + 1]\sigma)(k + \omega\sigma) \tag{5.9}$$

$$\mu_\chi = -16\omega\alpha(k + 2[\omega + 1]\sigma)(3k + 2[2\omega + 1]\sigma) \ . \tag{5.10}$$

As in sections 3 and 4, if the parameters μ_γ and μ_χ are not positive the solution will have bulk ghosts.

In contrast to the Einstein gravity solutions of section 4 the tensor and scalar perturbations ($\bar{\gamma}_{\mu\nu}$ and χ respectively) have different behaviour in the bulk. For example, if $f_\gamma \ll f_\chi$, then the tensor perturbations will be less localised on the brane than the scalar perturbations. In further contrast to the bulk field equations of the previous section, the parameters f_γ^2 and f_χ^2 may be negative. If this happens, the corresponding field equation will resemble that for a field in a spacetime with three timelike coordinates (or two if $\mu_{\gamma,\chi}$ is also negative). This will give rise to instabilities on the quantum level and, for some parameter ranges, on the classical level too. Either way, we need to have not just μ_γ and μ_χ positive, but also f_γ^2 and f_χ^2.

The brane junction conditions imply

$$2\mu_\gamma\partial_z\bar{\gamma}_{\mu\nu} + (\beta + 8\alpha[k - 2\sigma])\Box_4\bar{\gamma}_{\mu\nu} = -\frac{2\sqrt{\alpha}}{M_\star^2}\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\} \tag{5.11}$$

$$\mu_\chi\left(2(k - \sigma)\partial_z\chi - f_\chi^2\Box_4\chi\right) + 2k\left([3 + 2\omega]\beta + 8\alpha[\omega + 2][3k + 2\omega\sigma]\right)\Box_4\varphi = -k\frac{\sqrt{\alpha}}{M_\star^2}S \tag{5.12}$$

$$\mu_\chi\sigma\partial_z\chi - 3k(\beta + 16\alpha[k + \omega\sigma])\Box_4\chi + (4 + \beta[3k + 2\omega\sigma])\sigma\Box_4\varphi = 0 \tag{5.13}$$

with $M_\star^2 = M^3 e^{-2\phi_0}\sqrt{\alpha}$. If the coefficient of the $\Box_4\bar{\gamma}_{\mu\nu}$ term in the boundary condition (5.11) is negative, the system may be unstable. If $k < 0$ the situation will be similar to that in the previous section, and there will be a graviton tachyon mode. The same will be true in $k > 0$ and μ_γ is not too small. Alternatively if μ_γ is sufficiently small there will be no tachyon, but the effective four dimensional gravitational coupling will have the wrong sign. Similar considerations apply to the scalar modes.

If we take the negative choice of sign for the solution (5.1), then the requirement that there are no ghosts restricts σ to be less than $\sigma_* = [8\alpha(2\omega^2 + 7\omega + 6)]^{-1/2}$. In this range

$8\alpha[k-2\sigma] < 0$ and k is always negative. This branch is therefore unstable (for small β). As with the solution in section 4 the instability could be removed by including a large induced gravity term (i.e. taking $\beta > -8\alpha[k-2\sigma]$). We will consider this in section 6.

For the rest of this section we will take the positive sign in eq. (5.1). For this branch, k, μ_γ, μ_χ and f_χ^2 are all positive, and $f_\chi^2 \geq f_\gamma^2$. For f_γ^2 to be positive we require that $\sigma < \sigma_c$, where $\sigma_c = [4\alpha(\omega^2 + 2\omega + 3)]^{-1/2}$. To avoid tachyons (graviton and scalar) we also need $\beta + 8\alpha(k - 2\sigma) > 0$. If $\beta = 0$ this is true when $\sigma \leq \sigma_*$. Since $\sigma_* \leq \sigma_c$, the junction conditions provide a stronger constraint than the bulk effects.

The bulk graviton equation (5.5) is solved by

$$\bar{\gamma}_{\mu\nu}(p, z) \propto e^{(2k-\sigma)z} K_{2-\sigma/k} \left(f_\gamma p e^{kz}/k \right) . \tag{5.14}$$

The bulk scalar equation has a similar solution, but with f_γ replaced by f_χ. The bulk graviton perturbations are qualitatively similar those in the conventional brane world (see section 3). In fact we can get something similar to the Randall-Sundrum model (with a scalar field and induced gravity term) if we take $\sigma \to 0$. In particular f_χ and f_γ will be both equal to one.

The solution of the bulk field equation (5.14) implies that

$$\partial_z \bar{\gamma}_{\mu\nu} = -(p f_\gamma) \frac{K_{1-\sigma/k}(f_\gamma p/k)}{K_{2-\sigma/k}(f_\gamma p/k)} \bar{\gamma}_{\mu\nu} \tag{5.15}$$

on the brane. A similar relation holds for $\partial_z \chi$. By using asymptotic and series expansions of the above expression, we can determine the approximate small and large distance effective gravitational laws on the brane (just as we did in section 3). If $k > \sigma$ (as is the case for the solutions we are considering), then when p is small $\partial_z \bar{\gamma}_{\mu\nu} \approx -f_\gamma^2 p^2 \bar{\gamma}_{\mu\nu}/[2(k - \sigma)]$, and when p is large $\partial_z \bar{\gamma}_{\mu\nu} \approx -f_\gamma p \bar{\gamma}_{\mu\nu}$.

At large distances, when $p \ll k/f_\chi$, the junction conditions (5.11-5.13) imply that the four dimensional effective linearised gravity will be a tensor-scalar theory [see eqs. (3.10) and (3.11)] with

$$M_{\mathrm{Pl}}^2 = \left(\beta + \frac{2 - \mu_\gamma}{k - \sigma} \right) \frac{M_\star^2}{\sqrt{\alpha}} \tag{5.16}$$

$$M_\phi^2 = \left[(3 + 2\omega)(\beta/\sqrt{\alpha}) + 8\sqrt{\alpha}(\omega + 2)(3k + 2\omega\sigma) \right] M_\star^2 . \tag{5.17}$$

In this case the effective brane scalar field, $\tilde{\varphi}$, is just φ.

To avoid conflict with experimental measurements, we would like to have $M_{\mathrm{Pl}} \ll M_\phi$ [5]. For $\omega = -1$ (so $\sigma_* = \sigma_c$) and $\beta = 0$ this will be true when $f_\gamma \approx 0$, which occurs when $\sigma \approx \sigma_c$. We then have $M_\phi \sim M_\star$ and $M_{\mathrm{Pl}}/M_\phi \sim f_\gamma$. Thus a hierarchy between the effective scalar and tensor couplings has arisen, despite the fact that the fundamental, five-dimensional scalar and gravitational couplings are of the same order. This is possible because the scalar field and the higher gravity terms have affected the different parts of the perturbation in different ways. This only occurs when the model includes both these features. If Φ is turned off ($\sigma \to 0$), then $f_\gamma, f_\chi \to 1$ and both types of perturbation 'feel' the effects of the bulk in the same way. On the other hand if the higher gravity terms are dropped ($\alpha \to 0$), the above solution does not exist in the first place.

A similar effect can be obtained for other values of ω, but unfortunately the region of parameter space with $f_\gamma \approx 0$ ($\sigma \approx \sigma_c$) has $8\alpha[k - 2\sigma] < 0$, which implies that tachyons will be present.. This can be fixed by including an appropriate induced gravity term with coefficient $\beta = \beta_c = 8\alpha(1 + \omega)\sigma_c$. A hierarchy is then obtained, but at the price of a fine-tuning β.

At very short distances, when $p \gg k/f_\gamma$, the $\Box_4$ terms in the boundary conditions dominate the ∂_z terms. The resulting effective gravitational theory is again scalar-tensor. The Planck mass is then

$$M_{\rm Pl}^2 = [(\beta/\sqrt{\alpha}) + 8\sqrt{\alpha}(k - 2\sigma)]M_\star^2 \ . \tag{5.18}$$

This time the effective scalar mode is a combination of γ and φ. This gives a different relation between the two mass scales,

$$M_\phi^2 = 3M_{\rm Pl}^2 \left[1 + \omega\frac{M_{\rm Pl}^2}{M_\star^2}\left(\frac{3M_{\rm Pl}^2}{2M_\star^2} + \frac{\omega\mu_\gamma}{k\sqrt{\alpha}} + \frac{96\alpha(k + 2[1 + \omega]\sigma)}{(M_{\rm Pl}/M_\star)^2 + 16\sqrt{\alpha}(k + 2[1 + \omega]\sigma)}\right)^{-1}\right] \ . \tag{5.19}$$

For the parameter ranges we are considering, this gives $M_\phi^2 \leq 3M_{\rm Pl}^2$, even when σ is fine tuned to be near σ_c. The suppression of the scalar part of the gravity is lost at short distances. However the short distance constraints on Brans-Dicke gravity are weak, so this is not a problem if the length scale f_χ/k is of geographical size or smaller.

For $k/f_\gamma \gg p \gg k/f_\chi$, tensor perturbations have large distance behaviour, while scalar perturbations have small distance behaviour. This third regime is another new feature of this brane world mode which is not present in the standard brane world. $M_{\rm Pl}$ is again given by eq. (5.16) as above, but this time there is a different effective scalar, and

$$M_\phi^2 = 3M_{\rm Pl}^2 \left[1 + \frac{M_{\rm Pl}^2 k(\sigma - k)}{4M_\star^2\sqrt{\alpha}(k + 2[1 + \omega]\sigma)\omega\sigma^2}\left(2 + \frac{3kY^2}{\sigma - k + \omega\sigma Y}\right)^{-1}\right] \tag{5.20}$$

where

$$Y = 1 + \frac{M_{\rm Pl}^2(\sigma - k)}{8M_\star^2\sqrt{\alpha}(k + 2[1 + \omega]\sigma)\omega\sigma} \ . \tag{5.21}$$

As with the short distance gravity, we find $M_\phi^2 \leq 3M_{\rm Pl}^2$. As before, this need not be a problem unless the length scale f_γ/k is of astronomical size.

The above expressions for $M_{\rm Pl}$ and M_ϕ at the three different scales are algebraically rather complicated, and not particularly revealing. It is instructive to take $\beta = \beta_c$, $\sigma = \sigma_c - \sqrt{\alpha}\epsilon$, and then consider ϵ close to zero. In this case

$$M_{\rm Pl}^2 \approx \frac{8(\omega + 3)}{(-\omega)}M_\star^2\epsilon \tag{5.22}$$

at large and medium distances, and

$$M_{\rm Pl}^2 \approx \frac{8(\omega + 3)(\omega + 2)}{3}M_\star^2\epsilon \tag{5.23}$$

at short distances. At medium and short distances, it can be seen from eqs. (5.19) and (5.20) that M_ϕ^2 is approximately 3 times the corresponding $M_{\rm Pl}^2$ (recall $M_{\rm Pl} \ll M_\star$ when $\beta = \beta_c$ and $\sigma \approx \sigma_c$). At large distances

$$M_\phi^2 \approx \frac{4(\omega + 3)^2}{\sqrt{\omega^2 + 2\omega + 3}} M_\star^2 \ . \tag{5.24}$$

We see that Brans-Dicke gravity constraints could be satisfied (at large distance scales) by taking $\epsilon \lesssim 10^{-3}$. On the other hand, we see from eqs. (5.22) and (5.23) that there is at least a factor of 3 difference between $M_{\rm Pl}^2$ on medium and small distance scales. This may lead to conflict with observations. Of course, any problems could be avoided by taking the distance scale f_γ/k to be very small. In this case gravity on all scales at which accurate gravitational experiments have been performed would be given by eqs. (5.16) and (5.17).

6. Scalar Gravity

We saw in section 4 that if second order bulk curvature terms were added to a brane world solution with a negative warp factor, it could develop a tachyon mode. If a sufficiently large induced gravity term is added to the action, this instability will be removed. Having included such a term, the approximate gravitational law on the brane can be obtained (as with the other brane world solutions) by substituting the solutions to the linearised bulk equations (4.13,4.14) into the junction conditions (4.8,4.9).

At relatively small distances ($p \gg |k|$), we obtain a scalar-tensor theory of gravity (3.10,3.11) with

$$M_{\rm Pl}^2 = \left[(\beta/\sqrt{\alpha}) - 16(\omega + 2)\sqrt{\alpha}\sigma \right] M_\star^2 \tag{6.1}$$

and

$$M_\phi^2 = (4 + 3\omega) \frac{\mu_0 - \sigma(3 + 2\omega)M_{\rm Pl}^2}{(2 + \omega)\mu_0 - \sigma(4 + 3\omega)M_{\rm Pl}^2} M_{\rm Pl}^2 \ . \tag{6.2}$$

If $\omega = -1$ then $M_\phi = M_{\rm Pl}$. For other values of ω it appears to be possible to obtain $M_\phi \gg M_{\rm Pl}$ by fine-tuning the denominator of eq. (6.2), but unfortunately M_ϕ^2 is negative in this region of parameter space. Thus, for any choice of β, the above theory will not be suitable for describing gravity in our universe at astronomical distances.

Looking at the behaviour of the perturbations for small momenta (large distances), we see that the junction conditions (4.8,4.9) imply that $\bar\gamma_{\mu\nu}, \chi \sim S$, while (for $\omega \neq -1$) $\varphi, \gamma \sim S/p^2$. To leading order in p the tensor modes are negligible and so at large distances ($p \ll |k|$), we have a scalar theory of gravity. A model with similar behaviour was discussed in ref. [19].

By considering the motion of a test particle which is confined to the brane, we see that $d^2x^i/dt^2 \approx -\hat\Gamma_{00}^i$, and so (for static solutions) the gravitational potential is $V = (1/2)\gamma_{00}$. For the above scalar theory of gravity there is an order $1/p^2$ contribution to V from γ, and so a Newtonian potential is obtained. If on the other hand the particle is not confined to the brane, $d^2x^i/dt^2 \approx -\Gamma_{00}^i = -\hat\Gamma_{00}^i + kv_i$. Using the expression for v_i (4.4), we find that now only $\bar\gamma_{00}$ and χ contribute to V, and so no Newtonian potential is obtained. Therefore

any Newtonian potential (at large distances) is purely a result of the brane's curvature, and nothing to do with the bulk behaviour of the perturbations.

The large distance behaviour of the above model is also incompatible with astronomical observations. The addition of an induced gravity term to the solutions of section 4 may have solved the stability problem, but it still does not produce a theory which could represent our universe. If $\omega = -1$, we will instead have a non-standard, scalar-tensor, large distance gravitational law, with a non-Newtonian gravitational potential. We will leave the analysis of such a theory for future work.

The 'Gauss-Bonnet' solutions in section 5 with the negative sign choice in eq. (5.1) suffered from a similar instability to the above 'Einstein' solution. This can also be cured by adding a large induced gravity term. In general, these stabilised solutions will also behave as scalar gravity theories with Newtonian gravitational potentials at large distances. At short distances they will be approximately Brans-Dicke, with $M_{\rm Pl}$ and M_ϕ given by eqs. (5.18) and (5.19). In contrast to the above 'Einstein' solution, it does appear to be possible to fine-tune β so that $M_\phi \gg M_{\rm Pl}$ without having $M_\phi < 0$ (although we will not discuss the algebraic details here). In this case a hierarchy is obtained by fine-tuning the positive contribution to the scalar coupling from the induced action (2.3), so that it cancels the negative contribution from the other boundary terms (2.2). This contrasts with the hierarchy obtained in section 5, which arose from the properties of the bulk background solution.

As with the other 'Gauss-Bonnet' solutions discussed in section 5, these solutions also have a medium distance gravitational law which is different from the small and large distance laws. Since this solution branch has $f_\chi \leq f_\gamma$, the tensor modes will have short distance behaviour in this region, and the scalar modes will have large distance behaviour (the reverse of what happens for the other 'Gauss-Bonnet' solutions). Thus for $k/f_\chi \gg p \gg k/f_\gamma$ we will again have scalar-tensor gravity, but with different effective mass scales to the short distance theory.

7. Conclusions

In this paper we have investigated the effects of higher order curvature terms and a scalar field on linearised gravity in a brane world scenario. These two extensions of the conventional brane world give rise to many new features. The higher order gravity produces additional terms in the brane junction conditions. These resemble those produced by an induced gravity term, and allow gravity to be four dimensional at all length scales, weakening constraints on the model. Another, less desirable, consequence is the possibility of new instabilities. If the curvature is too high, corrections to the bulk equations result in the appearance of ghosts. If the extra contribution to the junction conditions has the wrong sign, the model will have tachyons.

Not surprisingly, the gravitational field equations for the quadratic theory have more solutions than the linear one. One of the two extra solution branches gives a better behaved brane world than the one obtained from the linear theory. In particular it is stable, and its

bulk space is warped in a similar way to the Randall-Sundrum model. The other solution branches are unstable when the higher order gravity effects are included.

The introduction of the scalar field alters how perturbations 'feel' the warping of the bulk spacetime. The various coefficients in the linearised field equations are altered by presence of the bulk scalar field and the higher gravity terms. Significantly the changes to the scalar and tensor perturbation equations are not the same, and so with some fine-tuning of the background solution, it is possible to suppress the scalar modes relative to the tensor modes. This does not happen unless we include both the scalar field *and* the higher gravity terms. The effective four-dimensional theory on the brane is Brans-Dicke gravity which, thanks to the suppression of the scalar modes, could be compatible with observational constraints. However it should be noted that the constraints on Brans-Dicke gravity were obtained by going beyond a linearised approximation of the theory. Since we have only considered a linearised theory in this paper, the actual constraints may be different.

The instabilities in the other solution branches can (in some cases) be removed by including sufficiently large induced gravity terms in the brane action. The resulting theory is then a standard four-dimensional gravity action, plus the brane world part of the model. Since Brans-Dicke theory is stable, it is not too surprising that the resulting combined theory will also be stable if the brane world part of it is small. The resulting brane gravity is approximately scalar-tensor at short distances. It appears that the scalar modes can be suppressed in some cases by tuning the size of the induced term to cancel the brane world contribution. A Newtonian potential is obtained at large distances, although (to leading order) the resulting theory has no tensor part.

Having included quadratic curvature terms in the theory it is very natural to ask whether even higher order terms should be included as well. If the solutions in this paper had small curvature, we could safely ignore higher terms. Inevitably this is not the case, since if the curvature were small, the effects of the quadratic terms would also be small, and our brane world solutions would not be significantly different from the usual brane world scenarios with scalar fields. Work on brane worlds with higher order gravity actions [10] (without scalar fields) suggests that these higher curvature terms will give qualitatively similar contributions to the linearised gravity as the quadratic ones. Hence the results obtained in this paper should remain qualitatively unchanged, although obviously the algebraic expressions for the different mass and length scales in the models will be different.

Acknowledgements

A. Appendix: Field Equations and Boundary Terms

Variations of the various terms in the actions (2.1) and (2.2) are given below. The results are derived for a D-dimensional manifold, $\mathcal{M}$, bounded by a hypersurface Σ. The non-standard convention that the normal of Σ points into $\mathcal{M}$ is used, so $\int_{\mathcal{M}} d^D x \sqrt{-g}\, \nabla^a F_a = -\int_{\Sigma} d^{D-1} x \sqrt{-h}\, n^a F_a$ (for a brane world, this is slightly easier to visualise). Note that there is an extra factor of two in the brane world boundary terms, since the brane is treated as the boundaries of each of the two bulk half-spacetimes. In the expressions below ξ is an arbitrary function. In the rest of the paper it was equal to $e^{-2\Phi}$.

$$
\delta \left\{ \int_{\mathcal{M}} d^D x \sqrt{-g}\, \xi R - 2 \int_{\Sigma} d^{D-1} x \sqrt{-h}\, \xi K \right\}
$$
$$
= \int_{\mathcal{M}} d^D x \sqrt{-g}\, \delta g^{ab} \left\{ \xi G_{ab} - \nabla_a \nabla_b \xi + g_{ab} \nabla^2 \xi \right\} + \delta\Phi \{ \partial_\Phi \xi R \}
$$
$$
- \int_{\Sigma} d^{D-1} x \sqrt{-h}\, \delta g^{ab} \left\{ \xi(K_{ab} - K h_{ab}) - \partial_n \xi h_{ab} \right\} + \delta\Phi \{ 2\partial_\Phi \xi K \} \tag{A.1}
$$

$$
\delta \int_{\mathcal{M}} d^D x \sqrt{-g}\, \xi (\nabla\Phi)^2
$$
$$
= \int_{\mathcal{M}} d^D x \sqrt{-g}\, \delta g^{ab} \left\{ \xi \nabla_a \Phi \nabla_b \Phi - \frac{\xi}{2} g_{ab} (\nabla\Phi)^2 \right\} + \delta\Phi \left\{ \partial_\Phi \xi (\nabla\Phi)^2 - 2\nabla^a \xi \nabla_a \Phi - 2\xi \nabla^2 \Phi \right\}
$$
$$
- \int_{\Sigma} d^{D-1} x \sqrt{-h}\, \delta\Phi \left\{ 2\xi \partial_n \Phi \right\} \tag{A.2}
$$

Before dealing with the quadratic curvature terms, it is useful to define the tensors

$$
J_{ab} = \frac{1}{3} \left(2K K_{ac} K^c{}_b + K_{cd} K^{cd} K_{ab} - 2K_{ac} K^{cd} K_{db} - K^2 K_{ab} \right) , \tag{A.3}
$$

$$
P_{abcd} = R_{abcd} + 2R_{b[c} g_{d]a} - 2R_{a[c} g_{d]b} + R g_{a[c} g_{d]b} , \tag{A.4}
$$

so $\mathcal{L}_{\mathrm{GB}} = P^{abcd} R_{abcd}$. The second order Lovelock Tensor [7] is then

$$
H_{ab} = P_a{}^{cde} R_{bcde} - \frac{1}{4} g_{ab} \mathcal{L}_{\mathrm{GB}} . \tag{A.5}
$$

$$
\delta \left\{ \int_{\mathcal{M}} d^D x \sqrt{-g}\, \xi \mathcal{L}_{\mathrm{GB}} - \int_{\Sigma} d^{D-1} x \sqrt{-h}\, 4\xi (J - 2\widehat{G}^{ab} K_{ab}) \right\}
$$
$$
= \int_{\mathcal{M}} d^D x \sqrt{-g}\, \delta g^{ab} \{ 2\xi H_{ab} + 4P_{acbe} \nabla^e \nabla^c \xi \} + \delta\Phi \{ \partial_\Phi \xi \mathcal{L}_{\mathrm{GB}} \}
$$
$$
- \int_{\Sigma} d^{D-1} x \sqrt{-h}\, \delta g^{ab} \Big\{ \xi \left(6J_{ab} - 2J h_{ab} - 4\widehat{P}_{acbe} K^{ce} \right)
$$
$$
+ 2\partial_n \xi \left(2\widehat{G}_{ab} + 2K_{ea} K^e{}_b - 2K K_{ab} + h_{ab}[K^2 - K_{cd} K^{cd}] \right)
$$
$$
+ 4 \left(2K_{c(a} D^c D_{b)} \xi - K D_a D_b \xi - K_{ab} D^2 \xi - h_{ab}[K_{ec} D^e D^c \xi - K D^2 \xi] \right) \Big\}
$$
$$
+ \delta\Phi \left\{ 4\partial_\Phi \xi (J - 2\widehat{G}^{ab} K_{ab}) \right\} \tag{A.6}
$$

$$\delta\left\{\int_{\mathcal{M}}d^Dx\sqrt{-g}\,\xi G_{ab}\nabla^a\Phi\nabla^b\Phi - \int_{\Sigma}d^{D-1}x\sqrt{-h}\,\xi(K_{ab}-Kh_{ab})D^a\Phi D^b\Phi\right\}$$

$$= \int_{\mathcal{M}}d^Dx\sqrt{-g}\,\delta g^{ab}\left\{\xi\Big(P_{acbe}\nabla^c\Phi\nabla^e\Phi + \frac{1}{2}G_{ab}(\nabla\Phi)^2 + \nabla_c\nabla_{(a}\Phi\nabla_{b)}\nabla^c\Phi - \nabla_a\nabla_b\Phi\nabla^2\Phi\Big)\right.$$

$$+ \nabla^c\Phi\nabla_{(a}\xi\nabla_{b)}\nabla_c\Phi + \nabla^c\xi\nabla_{(a}\Phi\nabla_{b)}\nabla_c\Phi - \nabla^2\Phi\nabla_{(a}\Phi\nabla_{b)}\xi - \nabla^c\Phi\nabla_c\xi\nabla_a\nabla_b\Phi$$

$$- \nabla_{(a}\Phi\nabla_{b)}\nabla_c\xi\nabla^c\Phi + \frac{1}{2}\left(\nabla_a\Phi\nabla_b\Phi\nabla^2\xi + (\nabla\Phi)^2\nabla_a\nabla_b\xi\right)$$

$$+ \frac{1}{2}g_{ab}\Big[\xi(\nabla^2\Phi)^2 - (\nabla\Phi)^2\nabla^2\xi + 2\nabla^c\xi\nabla_c\Phi\nabla^2\Phi$$

$$\left.- \xi\nabla_c\nabla_e\Phi\nabla^c\nabla^e\Phi + \nabla^c\Phi\nabla^e\Phi\nabla_c\nabla_e\xi - 2\nabla^c\xi\nabla^e\Phi\nabla_c\nabla_e\Phi\Big]\right\}$$

$$+ \delta\Phi G_{ab}\left\{\partial_\Phi\xi\nabla^a\Phi\nabla^b\Phi - 2\xi\nabla^a\nabla^b\Phi - 2\nabla^a\xi\nabla^b\Phi\right\}$$

$$- \int_{\Sigma}d^{D-1}x\sqrt{-h}\,\delta g^{ab}\left\{\xi\Big(\frac{1}{2}[(\partial_n\Phi)^2 - (D\Phi)^2][K_{ab}-h_{ab}K] + \partial_n\Phi(D_aD_b\Phi - h_{ab}D^2\Phi)\right.$$

$$+ 2K_{c(a}D^c\Phi D_{b)}\Phi - KD_a\Phi D_b\Phi - h_{ab}K^{ec}D_e\Phi D_c\Phi\Big)$$

$$\left.+ \partial_n\Phi\left[D_{(a}\Phi D_{b)}\xi - h_{ab}D^c\Phi D_c\xi\right] - \frac{1}{2}\partial_n\xi\left[D_a\Phi D_b\Phi - h_{ab}(D\Phi)^2\right]\right\}$$

$$+ \delta\Phi\left\{\xi(K^2 - K^{ab}K_{ab} - \widehat{R})\partial_n\Phi\right.$$

$$\left.+ \left(\partial_\Phi\xi D^a\Phi D^b\Phi - 2D^a\xi D^b\Phi - 2\xi D^a D^b\Phi\right)(K_{ab}-Kh_{ab})\right\}\qquad(A.7)$$

$$\delta\left\{\int_{\mathcal{M}}d^Dx\sqrt{-g}\,\xi(\nabla\Phi)^2\nabla^2\Phi + \int_{\Sigma}d^{D-1}x\sqrt{-h}\,\xi\partial_n\Phi\left(\frac{1}{3}(\partial_n\Phi)^2 + (D\Phi)^2\right)\right\}$$

$$= \int_{\mathcal{M}}d^Dx\sqrt{-g}\,\delta g^{ab}\left\{\xi\left[\nabla_a\Phi\nabla_b\Phi\nabla^2\Phi - 2\nabla_c\Phi\nabla_{(a}\Phi\nabla_{b)}\nabla^c\Phi + g_{ab}\nabla_c\nabla_e\Phi\nabla^c\Phi\nabla^e\Phi\right]\right.$$

$$\left.+ \frac{1}{2}(\nabla\Phi)^2\left[g_{ab}\nabla_c\xi\nabla^c\Phi - 2\nabla_{(a}\xi\nabla_{b)}\Phi\right]\right\}$$

$$\delta\Phi\left\{2\xi[R_{ab}\nabla^a\Phi\nabla^b\Phi + \nabla_a\nabla_b\Phi\nabla^a\nabla^b\Phi - (\nabla^2\Phi)^2] + 4\nabla_a\xi\nabla_b\Phi\nabla^a\nabla^b\Phi\right.$$

$$\left.+ (\partial_\Phi\xi(\nabla\Phi)^2 - 2\nabla_a\xi\nabla^a\Phi)\nabla^2\Phi + \nabla^2\xi(\nabla\Phi)^2\right\}$$

$$- \int_{\Sigma}d^{D-1}x\sqrt{-h}\,\delta g^{ab}\left\{-\xi\partial_n\Phi\left[\frac{1}{3}(\partial_n\Phi)^2 h_{ab} + D_a\Phi D_b\Phi\right]\right\}$$

$$+ \delta\Phi\left\{2\xi\left(2\partial_n\Phi D^2\Phi + K(\partial_n\Phi)^2 + K^{ec}D_e\Phi D_c\Phi\right) + 2\partial_n\Phi D^a\xi D_a\Phi\right.$$

$$\left.- \partial_\Phi\xi\partial_n\Phi\left(\frac{1}{3}(\partial_n\Phi)^2 + (D\Phi)^2\right) - \partial_n\xi[(\partial_n\Phi)^2 + (D\Phi)^2]\right\}\qquad(A.8)$$

$$\delta\left\{\int_{\mathcal{M}}d^Dx\sqrt{-g}\,\xi(\nabla\Phi)^4\right\}$$

$$= \int_{\mathcal{M}}d^Dx\sqrt{-g}\,\delta g^{ab}\left\{\xi(\nabla\Phi)^2\left(2\nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}(\nabla\Phi)^2\right)\right\}$$

$$+ \delta\Phi \left\{ \partial_\Phi \xi (\nabla\Phi)^4 - 4\nabla_a \xi \nabla^a \Phi (\nabla\Phi)^2 - 4\xi(\nabla\Phi)^2 \nabla^2\Phi - 8\xi\nabla_a\nabla_b\Phi\nabla^a\Phi\nabla^b\Phi \right\}$$

$$- \int_\Sigma d^{D-1}x \sqrt{-h}\, \delta\Phi \left\{ 4\xi\partial_n\Phi \left[(\partial_n\Phi)^2 + (D\Phi)^2 \right] \right\} \tag{A.9}$$

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